Name: ____________________________

Instructor: ________________________

Math 10560, Practice Exam 2.
March 20, 2024

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 1 hour and 15 min.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 16 pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	________
11.	________
12.	________
13.	________
14.	________
15.	________
16.	________
Total	________

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Multiple Choice

1.(6 pts.) Evaluate the improper integral

$$\int_4^\infty \frac{1}{(x-2)(x-3)}\,dx.$$

Using a partial fraction expansion $\int \frac{1}{(x-2)(x-3)}dx = \ln|\frac{x-3}{x-2}| + C.$

Therefore $\int_4^\infty \frac{1}{(x-2)(x-3)}dx = \lim_{t\to\infty} \ln|\frac{t-3}{t-2}| - \ln|\frac{1}{2}| = 0 + \ln 2.$

(a) $\ln 3$ (b) $\ln \frac{1}{2}$ (c) $\ln 2$

(d) the integral diverges (e) $3\ln 2$

2.(6 pts.) What can be said about the integrals

$$(i)\int_0^1 \frac{e^x}{x^2}\,dx;$$

$$(ii)\int_1^\infty \frac{\cos^2 x}{x^2}\,dx?$$

Integral (i) diverges by the Comparison Theorem since the integrand is greater than $\frac{1}{x^2}$.

Integral (ii) converges by the Comparison Theorem since the integrand is less than $\frac{1}{x^2}$.

(a) both (i) and (ii) converge

(b) both (i) and (ii) diverge

(c) (i) converges and (ii) diverges

(d) (i) diverges and (ii) converges

(e) neither integral (i) nor (ii) is improper

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3.(6 pts.) Which of the following is an expression for the arclength of the curve $y = \cos x$ between $x = \frac{-\pi}{2}$ and $x = \frac{\pi}{2}$?

The arclength formula gives the answer as $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + (-sinx)^2} dx$.

(a) $2 \int_0^{\frac{\pi}{2}} \sqrt{1 + 2\sin^2 x}\, dx.$

(b) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \sin^2 x}\, dx.$

(c) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + \sin^2 x}\, dx.$

(d) $\frac{\pi^2}{2}$

(e) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \cos^2 x}\, dx.$

4.(6 pts.) Consider the following sequences:

$(I)\ \left\{(-1)^n \frac{n^2-1}{2^n}\right\}_{n=1}^{\infty} \qquad (II)\ \left\{(-1)^n \frac{n^2-1}{2n^2}\right\}_{n=1}^{\infty} \qquad (III)\ \left\{(-1)^n n \ln(n)\right\}_{n=1}^{\infty}$

Which of the following statements is true?

(I): By applying L'Hospital's Rule to the function $f(x) = \frac{x^2-1}{2^x}$ we can see that $\lim_{x\to\infty} f(x) = 0$. Thus $\lim_{n\to\infty} \frac{n^2-1}{2^n} = 0$. But for $n \geq 1$,

$$\frac{n^2-1}{2^n} = \left|(-1)^n \frac{n^2-1}{2^n}\right|,$$

so the sequence (I) also converges to 0.

(II): $\lim_{n\to\infty} \frac{n^2-1}{2n^2} = 1/2$, so as n grows large, the expression $(-1)^n \frac{n^2-1}{2n^2}$ oscillates between values close to $+1/2$ (when n is even) and values close to $-1/2$ (when n is odd). Thus the sequence (II) diverges.

(III): As $n \to \infty$, $n\ln(n)$ grows arbitrarily large. The factor of $(-1)^n$ in sequence (III)

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makes the values oscillate between positive values of large magnitude and negative values of large magnitude. Thus the sequence (III) diverges.

(a) Sequences I and II converge but sequence III diverges.

(b) All three sequences converge.

(c) Sequences II and III converge but sequence I diverges.

(d) All three sequences diverge.

(e) Sequence I converges but sequences II and III diverge.

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5.(6 pts.) Find the sum of the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^{n+1}}{3^n}.$$

This is a geometric series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots = \begin{cases} \text{converges to} \quad \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

(technically we should check if a_{n+1}/a_n is a constant r in order to check this.) We can identify a by calculating the first term with a_1. When $n = 1$, we get

$$a = a_1 = \frac{(-1)^1 2^{1+1}}{3^1} = -\frac{2^2}{3}.$$

When $n = 2$, we get

$$ar = a_2 = \frac{(-1)^2 2^{2+1}}{3^2} = \frac{2^3}{3^2}.$$

Now we have

$$r = \frac{a_2}{a_1} = \left(\frac{2^3}{3^2}\right) / \left(-\frac{2^2}{3}\right) = -\left(\frac{2^3}{3^2}\right)\left(\frac{3}{2^2}\right) = -\frac{2}{3}.$$

This means $a = -\frac{4}{3}$ and $r = -\frac{2}{3}$. Then $|r| < 1$ so the series converges to

$$\frac{a}{1-r} = \frac{-\frac{4}{3}}{1 - \frac{-2}{3}} = -\frac{4}{5}$$

(a) This series diverges. (b) $-\frac{4}{5}$ (c) $-\frac{3}{5}$

(d) $\frac{4}{5}$ (e) $\frac{3}{5}$

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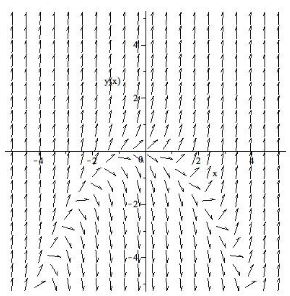
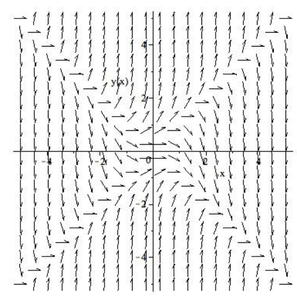
6.(6 pts.) Which of the following gives the direction field for the differential equation

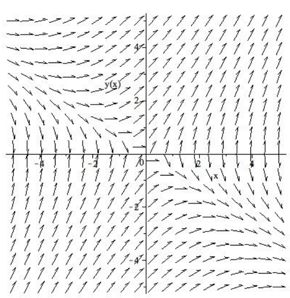
$$y' = y^2 - x^2$$

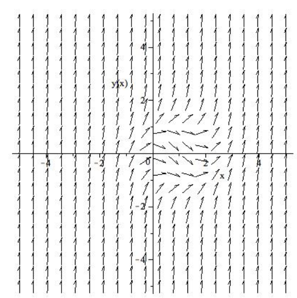
Note the letter corresponding to each graph is at the lower left of the graph.

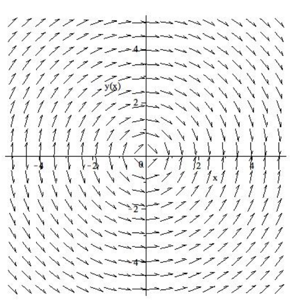
For points on the line $y = x$, we must have $y' = 0$. Also for points on the line $y = -x$, we must have $y' = 0$. Hence along both diagonals of the plane, we must have $y' = 0$ and the answer must be (b).

(a)


(b)


(c)


(d)


(e)


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7.(6 pts.) Use Euler's method with step size 0.1 to estimate $y(1.2)$ where $y(x)$ is the solution to the initial value problem

$$y' = xy + 1 \qquad y(1) = 0.$$

$$x_0 = 1, \quad y_0 = 0$$
$$x_1 = x_1 + h = 1.1, \quad y_1 = y_0 + h(x_0 y_0 + 1) = 0 + (0.1)(1 \cdot 0 + 1) = 0.1$$
$$x_2 = x_1 + h = 1.2, \quad y_2 = y_1 + h(x_1 y_1 + 1) = 0.1 + (0.1)((1.1)(0.1) + 1)$$
$$= 0.1 + 0.1(0.11 + 1) = 0.1 + 0.1(1.11) = 0.1 + 0.111 = 0.211$$

(a) $y(1.2) \approx .112$ (b) $y(1.2) \approx .211$ (c) $y(1.2) \approx .101$

(d) $y(1.2) \approx .201$ (e) $y(1.2) \approx .111$

8.(6 pts.) Find the solution of the differential equation

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1+x^2}$$

with initial condition $y(0) = 0$.

Separating variables here gives $\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{1+x^2}$.
Solving this gives $\arcsin y = \arctan x + C$ and substituting $y(0) = 0$ we find $C = 0$.
Therefore $y = \sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$.

(a) $y = \frac{x}{1+x}$ (b) $y = \frac{1}{\sqrt{1+x^2}}$ (c) $y = \frac{x}{\sqrt{1+x^2}}$

(d) $y = \frac{x}{1+x^2}$ (e) $y = \frac{x^2}{\sqrt{1+x^2}}$

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9.(6 pts.) Find a general solution, valid for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, of the differential equation

$$\frac{dy}{dx} - (\tan x)y = 1.$$

The integrating factor here is $I = e^{\int -\tan x dx} = e^{\ln(\cos x)} = \cos x$ and so a general solution is given by $y \cos x = \int \cos x dx = \sin x + C$. Therefore $y = \dfrac{\sin x + C}{\cos x}$.

(a) $y = \dfrac{x + \sin x + C}{\cos x}$ (b) $y = \dfrac{x + \sin x + C}{\sin x}$ (c) $y = \dfrac{\sin x + C}{\cos x}$

(d) $y = \tan x + \cos x + C$ (e) $y = \dfrac{\cos x + C}{\sin x}$

10.(6 pts.) A tank contains 1000 liters of water. Brine that contains 0.5 kg of salt per liter of water is added at a rate of 5 liters per minute. The solution is kept thoroughly mixed and drains from the tank at a rate of 5 liters per minute. What's the amount of salt after 3 hours and twenty minutes?

Let $y(t)$ denote the amount of salt in the tank after t minutes. We have $y(0) = 0$ and we wish to find $y(200)$.

We get a differential equation from

$$\frac{dy(t)}{dt} = \{\text{Salt in}\} - \{\text{Salt out}\} = \left[(.5) \times 5 - \frac{y(t)}{1000} \times 5\right] \text{ kg./min.}$$

Thus to find $y(t)$ we must solve a first order linear equation:

$$\frac{dy}{dt} + \frac{1}{200}y = 2.5.$$

The integrating factor is $I(t) = e^{\int (1/200)dt} = e^{t/200}$.

Multiplying the differential equation by $I(t)$, we get

$$e^{t/200}\frac{dy}{dt} + e^{t/200}\frac{1}{200}y = 2.5e^{t/200} \quad -> \quad \frac{d(e^{t/200}y)}{dt} = 2.5e^{t/200}.$$

Thus

$$e^{t/200}y = 2.5\int e^{t/200}dt = 500e^{t/200} + C.$$

Therefore $y = 500 + Ce^{-t/200}$. $y(0) = 0$ gives $C = -500$ and

$$y(t) = 500(1 - e^{-t/200}) \quad -> \quad y(200) = 500(1 - \frac{1}{e}).$$

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(a) $500(1 - e)$

(b) $500(1 - \frac{2}{e^3})$

(c) 500

(d) $500(1 - \frac{1}{e})$

(e) $500(1 - \frac{1}{e^2})$

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Partial Credit

You must show your work on the partial credit problems to receive credit!

11.(10 pts.) Calculate the arc length of the curve if $y = \frac{x^2}{4} - \ln(\sqrt{x})$, where $2 \leq x \leq 4$.

Solution: Recall

$$L = \int_a^b \sqrt{1 + (y')^2} dx.$$

Note

$$y' = \frac{x}{2} - \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2}\Big(x - \frac{1}{x}\Big).$$

Thus

$$1 + (y')^2 = 1 + \frac{1}{4}\Big(x - \frac{1}{x}\Big)^2 = 1 + \frac{1}{4}\Big(x^2 - 2x\frac{1}{x} + \frac{1}{x^2}\Big) = 1 + \frac{1}{4}\Big(x^2 - 2 + \frac{1}{x^2}\Big)$$
$$= 1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2} = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \frac{1}{4}\Big(x^2 + 2x\frac{1}{x} + \frac{1}{x^2}\Big) = \frac{1}{4}\Big(x + \frac{1}{x}\Big)^2.$$

Therefore

$$L = \int_2^4 \sqrt{1/4(x + 1/x)^2} dx = \int_2^4 \frac{1}{2}\Big(x + \frac{1}{x}\Big) dx = \frac{1}{2}\left[\frac{x^2}{2} + \ln x\right]_2^4 = 3 + \frac{1}{2}\ln 2.$$

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12.(10 pts.) (a) Circle the letter below alongside the trapezoidal approximation to

$$\ln 3 = \int_1^3 \frac{1}{x}dx \quad \text{using} \quad n = 8$$

$\boxed{A}$ $\int_1^3 \frac{1}{x}dx \approx \frac{1}{8}\left[1+2\left(\frac{4}{5}\right)+2\left(\frac{2}{3}\right)+2\left(\frac{4}{7}\right)+2\left(\frac{1}{2}\right)+2\left(\frac{4}{9}\right)+2\left(\frac{2}{5}\right)+2\left(\frac{4}{11}\right)+\left(\frac{1}{3}\right)\right]$

B $\int_1^3 \frac{1}{x}dx \approx \frac{1}{12}\left[1+4\left(\frac{4}{5}\right)+2\left(\frac{2}{3}\right)+4\left(\frac{4}{7}\right)+2\left(\frac{1}{2}\right)+4\left(\frac{4}{9}\right)+2\left(\frac{2}{5}\right)+4\left(\frac{4}{11}\right)+\left(\frac{1}{3}\right)\right]$

C $\int_1^3 \frac{1}{x}dx \approx \frac{1}{8}\left[1+\left(\frac{4}{5}\right)+\left(\frac{2}{3}\right)+\left(\frac{4}{7}\right)+\left(\frac{1}{2}\right)+\left(\frac{4}{9}\right)+\left(\frac{2}{5}\right)+\left(\frac{4}{11}\right)+\left(\frac{1}{3}\right)\right]$

(b) Recall that the error E_T in the trapezoidal rule for approximating $\int_a^b f(x)dx$ satisfies

$$\left|\int_a^b f(x)dx - T_n\right| = |E_T| \leq \frac{K(b-a)^3}{12n^2}$$

whenever $|f''(x)| \leq K$ for all $a \leq x \leq b$.

Use the above error bound to determine a value of n for which the trapezoidal approximation to $\ln 3 = \int_1^3 \frac{1}{x}dx$ has an error

$$|E_T| \leq \frac{1}{3}10^{-4}.$$

$$f(x) = \frac{1}{x}, \qquad f'(x) = \frac{-1}{x^2}, \qquad f''(x) = \frac{2}{x^3}$$

Since $|f''(x)| = \frac{2}{x^3}$ is decreasing on the interval $1 \leq x \leq 2$, we have $|f''(x)| \leq f''(1) = 2$ for $1 \leq x \leq 2$. Hence, we can use $K = 2$ in the error bound above.

For the trapezoidal approximation T_n, we have

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{2(3-1)^3}{12n^2} = \frac{16}{12n^2} = \frac{4}{3n^2}$$

If we find a value of n for which $\frac{1}{3}10^{-4} \geq \frac{4}{3n^2}$, then we will have $|E_T| \leq \frac{1}{3}10^{-4}$.

$$\frac{1}{3}10^{-4} \geq \frac{4}{3n^2} \quad \rightarrow \quad n^2 \geq 4 \cdot 10^4 \quad \rightarrow \quad n \geq 2 \cdot 10^2 = 200$$

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13.(10 pts.) Find the family of orthogonal trajectories to the family of curves given by

$$y = kx^2.$$

$$\frac{dy}{dx} = 2kx$$

For the family of curves given above

$$y = kx^2 \text{ giving } k = \frac{y}{x^2}$$

Thus this family of curves satisfy the differential equation

$$\frac{dy}{dx} = 2\frac{y}{x^2}x = 2\frac{y}{x}.$$

Now using the fact that the product of the derivatives of two orthogonal curves meeting at a point must equal -1, we get that the orthogonal trajectories satisfy the differential equation

$$\frac{dy}{dx} = \frac{-x}{2y}.$$

Separating the variables, we get

$$2ydy = -xdx$$

and

$$2\int ydy = -\int xdx, \quad \text{or} \quad y^2 = \frac{-x^2}{2} + C.$$

Hence our family of orthogonal trajectories is a family of curves of the form

$$y^2 + \frac{x^2}{2} = C,$$

a family of ellipses.

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14.(10 pts.) (10.3) A tank contains 5,000 liters of brine with 10 kg of dissolved salt. Pure water enters the tank at a rate of 50 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. Let $y(t)$ denote the amount of salt in the tank after t minutes.

Find a formula for $y(t)$.

We have $y(0) = 10$. The amount of liquid in the tank is $5000L$ at all times, giving a concentration of

$$\frac{y(t)}{5000}\frac{kg}{L} \quad \text{and} \quad \frac{dy}{dt} = -\frac{y(t)}{100}\frac{kg}{min} \quad \text{or} \quad \frac{dy}{dt} = -\frac{y}{100}$$

We can solve this equation by seperating variables:

$$\int \frac{1}{y} dy = -\frac{1}{100}\int dt$$

this gives

$$\ln y = -\frac{t}{100} + C.$$

Using the initial value condition $y(0) = 10$, we get $\ln 10 = C$ and

$$\ln y = -\frac{t}{100} + \ln(10).$$

Thus we get

$$\ln(\frac{y}{10}) = -\frac{t}{100} \quad \text{and} \quad \frac{y}{10} = e^{-t/100}$$

or

$$y(t) = 10e^{-t/100}kg.$$

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15.(10 pts.) **Extra Problem for Practice**
Solve the initial value problem

$$xy' + xy + y = e^{-x}$$
$$y(1) = \frac{2}{e}.$$

Solution: This is a linear differential equation. Since it can be reduced to the form

$$y' + \Big(1 + \frac{1}{x}\Big)y = \frac{e^{-x}}{x},$$

an integrating factor is

$$I(x) = e^{\int (1+\frac{1}{x})dx} = e^{x + \ln x} = xe^x.$$

Multiply both sides of the differential equation by $I(x)$ to get

$$xe^x y' + y(x+1)e^x = 1,$$

and hence

$$(xe^x y)' = 1.$$

Integrate both sides to obtain

$$xe^x y = x + C,$$

or

$$y = e^{-x}\Big(1 + \frac{C}{x}\Big).$$

Using the initial value, we have

$$y(1) = \frac{2}{e} = \frac{1}{e}(1 + C), \qquad C = 1.$$

Hence

$$y = e^{-x}\Big(1 + \frac{1}{x}\Big).$$

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16.(10 pts.) **Extra Problem for Practice**

Solve the initial value problem
$$\begin{cases} x^2y' + 2xy = 1, \\ y(1) = 2. \end{cases}$$

We put the equation in standard form:

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x^2}.$$

The integrating factor is given by

$$I(x) = e^{\int (2/x)dx} = e^{2\ln|x|} = x^2.$$

Multiplying by the integrating factor, we get

$$x^2\frac{dy}{dx} + 2xy = 1.$$

Therefore

$$\frac{d}{dx}x^2y = 1 \quad \text{and} \quad x^2y = \int 1dx = x + C$$

Dividing both sides by x^2, we get

$$y = \frac{x+C}{x^2}.$$

The initial condition gives $y(1) = 2$ or $1 + C = 2$ and $C = 1$. Hence

$$y = \frac{x+1}{x^2}.$$

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The following is the list of useful trigonometric formulas:

Note: $\sin^{-1} x$ and $\arcsin(x)$ are different names for the same function and $\tan^{-1} x$ and $\arctan(x)$ are different names for the same function.

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x-y) + \sin(x+y)\big)$$

$$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\int \sec\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\int \csc\theta = \ln|\csc\theta - \cot\theta| + C$$

$$\csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$